***Constitutive relations (CR) for anisotropic optically active media***

*Razvigor Ossikovski and Oriol Arteaga*

***Phenomenological constitutive relations*** containing spatial derivatives of the fields (Aleksandrov, Landau *–* Lifshitz, Fedorov, Yariv – Yeh, Huard; attributed to Gibbs),



where  and  are the electric and magnetic optical activity (OA) third-rank tensors. These CR physically describe natural optical activity as originating from the *spatial dispersion phenomenon*. Gibbs CR lead to Fedorov CR if  and  (*antisymmetry with respect to the second pair of indices*), as well as to Born – Landau CR if  (*antisymmetry with respect to the outer pair of indices*) *and* .

***Constitutive relations*** *(CR) used in the literature:*

1. *Tellegen* (both direct and reciprocal space forms)



The *optical activity tensor* **** (as used in Konstantinova’s paper, for instance) is real; its symmetry properties depend on the crystal class. The permittivity and permeability tensors **** and **µ** are symmetric for nonabsorbing media (**** is Hermitian in the presence of external magnetic field; **µ** is Hermitian for a ferromagnet) (Post’s book).

Tellegen CR are particularly well suited for the 4×4 matrix formalisms of light propagation in anisotropic (and optically active) media and can be shown to be *strictly* equivalent to Fedorov CR (see below) for *plane monochromatic waves*.

1. *Fedorov* (direct / reciprocal space forms); derived from the Gibbs CR involving spatial derivatives (Bokut – Serdyukov – Fedorov paper; Aleksandrov paper)





where **m** = *n* **u** is the *refraction vector*. Also, **m**× = *n* **u**× is the *refraction matrix* where the antisymmetric matrix **u**× (dual to the vector product) is



Fedorov CR (together with its “derivatives”, Born - Landau CR and Bokut’s - Serdyukov CR) formally coincide with those of *optically inactive* anisotropic media, through effective (curl - containing) permittivity and permeability tensors  and ,



The two tensors  and  are jointly affected by the presence of OA (expressed through ****1, even if the medium is non-magnetic (**µ**1 = **I**). In the absence of OA,  and . The Fedorov *OA tensors* **** and ****1 are real. The tensor ****1 “includes” the explicit spectral dependence. The energy density is



and is the only one that formally coincides with that for an *optically inactive* (anisotropic) medium,



For *plane monochromatic waves* (i.e., for the *reciprocal space* forms), Tellegen and Fedorov CRs can be shown to be equivalent (Barkovsky – Borzdov paper; see the end of the document). In particular, this means that for *plane monochromatic waves* Tellegen, Fedorov, (complete) Born - Landau and (complete) Bokut’ – Serdyukov CRs (see below) can be looked upon as deriving from the general, spatial dispersion, Gibbs CR.

Relationships between Fedorov and Tellegen CRs (Barkovsky-Borzdov paper),



and vice versa,



(The first two pairs of approximate equalities are valid to the first order in the OA; the last ones, to the second order.) The respective constitutive tensors are

 and 

(both Hermitian for non-dissipative media). Fedorov constitutive tensor is block-diagonal. The Tellegen constitutive tensor is a *material tensor* in the strict sense since it is independent of the direction of propagation (i.e. does not contain ).

1. *Condon* (direct / reciprocal space forms); involve time derivatives of the fields rather than spatial ones

 

The Condon OA tensor **** is real.

For *plane monochromatic waves* (i.e., for the *reciprocal space* forms), Tellegen and Condon CR are seen to be *strictly* equivalent through simple identification. The constitutive tensor is



(of form similar to Tellegen’s one).

1. *“Classic” Born – Landau* (direct / reciprocal space forms); historically derived from Gibbs CR involving spatial derivatives (Landau – Lifshitz and Born books)





so that



and the permeability tensor **µ** does not change. OA modifies the permittivity tensor only.

Here, **** is the *optical activity tensor*,



is the *reduced gyration tensor*,



is the *gyration tensor* and



is the *gyration vector*. These CR are first-order approximations in the OA.

The “classic” Born – Landau CR are of the same formal form as those of optically inactive anisotropic or isotropic media and furthermore, the permeability tensor **µ** is unaffected by the OA. The Born – Landau OA tensor **** is real. The gyration tensor **g** “includes” both the explicit spectral and refractive index dependencies. Only the component **G***u* of the gyration vector **G** along **u** matters (to the first order in the OA) because of  (Gauss’ law), so that  can be replaced by  in the constitutive relation. Consequently, only the symmetric part **g***s* of **g** matters (to the second order in OA) since  and therefore **g** can be assumed symmetric. (The optical activity tensor  can likewise be assumed symmetric.)

The energy density is



and the Poynting vector,



Energy is conserved to the second order in the OA (*w* and **S** are the corresponding quantities for an optically inactive medium). The constitutive tensor is



(of block-diagonal form similar to Fedorov’s one).

“Classic” Born – Landau and Fedorov CR can be shown to be equivalent to the first order in the OA (Bokut - Serdyukov paper).

*Strict* equivalence between *complete* Born – Landau CR (see later), Fedorov CR (Bokut - Serdyukov paper) and Tellegen CR,



or



and conversely



or



Since **** can be assumed symmetric within “classic” Born – Landau CR, the equivalence can be applied *only to the symmetric part* of ********. If **µ** = **I** (non-magnetic medium) then one further gets for the symmetric part *****sp* of the optical activity tensor **** from Tellegen CR (1) in terms of the *OA tensor* ****



and conversely



Equivalently, expression of the symmetric part *****sp* of **** in terms of the *gyration tensor* ,



and conversely,



where *n* is the refractive index along the wave normal, obtained from the equation of normals of *the anisotropic medium without OA* (see later). If furthermore the product (*anisotropy*) × (*optical activity*) is neglected that is, the equation of normals is solved without the OA contribution, then *n* is just the mean refractive index (agreement with Konstantinova’s paper).

Consequently, from an experiment described in terms of “classic” Born – Landau CR, one can determine only the symmetric part of the optical activity tensor **** from Tellegen CR (or of the tensor **** from Fedorov CR).

1. *“Classic” Bokut’ - Serdyukov* (direct / reciprocal space forms); the magnetic permeability, instead of the dielectric permittivity, is modified only; CR dual to the Born – Landau CR:





so that



and the permittivity tensor **** does not change.

Here, *****m* is the *magnetic optical activity tensor*, is the *reduced magnetic gyration tensor*,  is the *magnetic gyration tensor* and **G***m* = **g***m* **u** is the *magnetic gyration vector*. Bokut’ – Serdyukov CR can be shown to be approximately equivalent (to the first order in the OA) to Fedorov CR and to the Born – Landau CR (Bokut - Serdyukov paper).

*Strict* equivalence between *complete* Bokut’ - Serdyukov CR, Fedorov CR (Bokut - Serdyukov paper) and Tellegen CR (see later):



or



and conversely



or



One gets for the symmetric part *****sp* of the optical activity tensor **** from Tellegen CR in terms of the magnetic optical activity tensor *****m* (assumed symmetric),



and conversely



Equivalently, the symmetric part *****sp* of **** in terms of the magnetic gyration tensor **,**



and conversely,



where *n* is the mean refractive index if the product (*anisotropy*) × (*optical activity*) has been neglected (that is, is assumed). Note that the permeability tensor **µ** can be arbitrary (*magnetic*medium).

Consequently, from an experiment described in terms of “classic” Bokut’ – Serdyukov CR, one can determine only the symmetric part of the optical activity tensor **** from Tellegen CR (or of the tensor ****1 from Fedorov CR).

The constitutive tensor is



(block-diagonal; dual to Born – Landau one).

Relation with Born – Landau CR for a ***non-magnetic*** medium (**µ** = **I**):

 

and conversely



where ** = *n*2 is the mean dielectric constant. The product (*anisotropy*) × (*optical activity*) has been neglected ( has been assumed).

1. *Complete Born – Landau* (direct / reciprocal space forms); derived from the Fedorov CR involving spatial derivatives (curls) (after extending Bokut’ – Serdyukov paper); *strictly* equivalent to Fedorov (and Tellegen) CR,





where the *gyration tensor* **g** and *gyration vector* **G** = **g u** are as defined before, see “classic” Born – Landau CR , and



is the *second-order gyration tensor* and



is the *gyration matrix* associated with **g”**. Note that the second-order gyration tensor (and the gyration matrix associated with it) is strictly expressible in terms of the gyration and permittivity tensors, so that it is not independent, but rather appears as a second-order correction.

Both the second-order gyration tensor **g**” and the gyration matrix **** are symmetric. However, the gyration tensor **g** is not assumed symmetric anymore and has the same symmetries as the optical activity tensor ****from Tellegen CR or ****1 from Fedorov CR (see Post’s book and Fedorov’s papers),



(Here, ***a*** ≡ ****, ****1 or **g**; the symmetries depend on the crystal class.) The relations between the two tensors are as before (assuming a non-magneic medium, **µ** = **I**),



and



to the second order in the optical activity, so that



for the second-order gyration tensor **g**”. The last expression is valid if the product (*anisotropy*)×(*optical activity*) is neglected. When the gyration vector **G** is evaluated, one can use only the symmetric part of **g**, like before. The constitutive tensor is



(block-diagonal; of form similar to Born-Landau one).

Since the gyration matrix **** is symmetric, it appears as a second-order modification of the dielectric tensor ****, i.e. **** = ****. If the gyration vector **G** vanishes (**G** = **0**) which occurs for an antisymmetric gyration tensor **g**, then the medium exhibits no rotatory power (because of **G** = **0**) and the optical activity affects its dielectric tensor **** by modifying its linear birefringence to the second order only. This explains the experimental difficulties in evidencing optical activity in crystals with antisymmetric gyration (and optical activity) tensor(s) in normal transmission experiments. Modification of the linear birefringence (in rad/m) *in normal transmission*,



and



The circular birefringence CB (or rotatory power) is obtained as before (if **G** ≠ **0**). Most generally, for these crystals with antisymmetric gyration (and optical activity) tensors distinction must be made between rotatory power and optical activity, the latter manifesting itself as linear birefringence (Fedorov’s paper).

Energy density



and Poynting vector



The last terms in the above expressions can be regarded as second-order corrections to the respective quantities of the “classic” Born - Landau CR, necessary for the conservation of energy.

In summary, ***natural* optical activity** can be formally described either by (1.) Tellegen CR, (2.) Fedorov CR, (3.) Born – Landau CR (“classic”, or complete) and (4.) Bokut’ - Serdyukov CR (“classic”, or complete). All four descriptions are equivalent (except that the “classic” versions are first order approximations of the complete ones). By using (1.), OA is described by introducing the OA tensor in the off-diagonal blocks of the otherwise block-diagonal constitutive tensor, whereas by using (1.-4.) OA is described by modified permittivity and permeability tensors including curls (the constitutive tensor remains block-diagonal). The interpretation of the relationships between various descriptions depends on the choice of “fundamental” CR from which the rest of the CRs are supposed to be derived: (*a*.) if Tellegen CR are assumed fundamental, then both permittivity and permeability tensors of the rest of the CR change ; if (*b*.) if Fedorov CR are chosen, then both permittivity, permeability and optical activity tensors in Tellegen CR change.

***Description of the Faraday effect******and MOKE*** *(optical activity induced by an external magnetic field, evidenced in transmission or reflection):*

1. *Born – Landau CR*: (traditional treatment) If the medium is anisotropic (of a non-cubic crystal class), then the gyration vector **G** depends linearly on the external magnetic field (or induction),  where **f** and **** are material tensors (Landau – Lifshitz¸ Yariv – Yeh books). If the medium is isotropic (or a cubic crystal), the tensors are proportional to the identity (become scalars). Since **µ** = **I** (*non-magnetic* medium) is usually assumed (without being absolutely necessary), the Faraday effect thus described is called “dielectric” (Post’s book).
2. *Bokut’ – Serdyukov CR*: dual to the Born – Landau (traditional) treatment; more suitable for *magnetic* media. It is the magnetic permeability tensor **µ** (not the electric permittivity one, ****) that is modified to the first order by the external magnetic field, so that the medium can be *magnetic* (e.g., a ferrite). Consequently, the Faraday effect is termed “magnetic”, unlike the “dielectric” one modeled with Born – Landau CR (Post’s book). One has  where **f***m* and *****m* are material tensors (in full analogy with Landau – Lifshitz¸ Yariv – Yeh books). If the medium is isotropic (or a cubic crystal), the tensors are proportional to the identity (become scalars). The rest of the relations are fully analogous to those of Born – Landau CR. For *non-magnetic* media (**µ** = **I**) there is no qualitative difference between models (*b*) and (*a*). (The relation between electric and magnetic tensors **g** and **g***m* for that case is given above.)

In summary, “classic” Born – Landau and Bokut’ – Serdyukov CR *fully and exactly* describe *induced OA* (unlike *natural* one, approximately described to the first order in normal transmission only).

***Maxwell curl equations*** (direct / reciprocal space; CGS units):

  

where



is the wavevector and **m** = *n* **u** is the *refraction vector* (, called the *refraction matrix*, is the antisymmetric matrix dual to the vector product ). (The common phase in the reciprocal space is of the form ). In matrix form (for *plane monochromatic waves*),



The conservation of energy is expressed through



where **S** is the Poynting vector and *w* is the energy density.

***Generalized constitutive relations and* *eigenmatrices*** for the propagation of the electromagnetic field vector **F**, resulting from the subtraction of the Maxwell equations (in matrix form) from the generalized CR (Barkovsky – Borzdov paper):

1. Tellegen CR,

 

1. Fedorov CR,

 

1. Complete Born - Landau CR,

 

1. Complete Bokut’ - Serdyukov CR (dual to complete Born - Landau CR),

 

The determinants |**A**(…)| of the 6×6 eigenmatrices lead to the general equations of normals |**A**(…)| = 0 for the refractive indices *n*1,2 . The association of *n*1,2 with the eigenvectors belonging to the null space of *n*1,2  - the 3×1 part of the electric field **E** - leads to the Jones matrix in *bulk propagation* (which is experimentally equivalent to *normal transmission through a parallel slab with neglected interfaces*). In bulk propagation one takes into account only the eigenvalues (the refractive indices) and disregards the eigenvectors (the propagation modes).

To pass from *normal* to *oblique* incidence, one must replace  by  where ,



(**0 is the angle of incidence in the entrance medium of refractive index *n*0). No more 2×2 dimension reduction for the inductions takes place (see the next section).

Knowing the four refractive indices and the two sets of eigenvectors of **E** and **H** the 4×4 formalisms (Berreman and Yeh) can be constructed: *the two approaches, “old-fashioned” and 4*×*4 matrix, are equivalent*. More specifically, the resolution of the equation of normals and the subsequent determination of the null space of the eigenmatrices is mathematically equivalent to the diagonalization of the Berreman and Yeh propagation matrices. The first step provides the eigenvalues (the refractive indices) while the second one yields the eigenvectors (the polarization modes).

***Transformations of the eigenequations***

1. *Transformed* Tellegen eigenmatrix (Gaussian-upper block elimination, applicable if |**µ**| ≠ 0),



The eigenvalues and the eigenvectors for the electric field **E** *are invariant* by the transformation, so that they can be both determined from the 3×3 eigenmatrix . The equation of normals



which follows from  provides the refractive indices *n*1,2. The eigendimension effectively reduces from 6 to 3. The algebraic equation is quadratic for *n*2 (biquadratic for *n*) (since Hermitian transposition changes *n* into –*n* while maintaining the determinant unchanged, provided that  and ). The null spaces of the electric field eigenmatrices  provide the respective eigenvectors. The eigenvector components are given by the cofactors associated with any matrix row (no diagonalization is necessary!).

Alternatively, if |****| ≠ 0 then (Gaussian-lower block elimination)



and the equation of normals for the magnetic field is



since . The two transformations are dual due to the duality of the electric and magnetic fields.

When |**µ**| ≠ 0 *and* |****| ≠ 0, the refractive indices *n*1,2 can be obtained from anyone of the two equations of normals, electric or magnetic; their solutions coincide. The eigenvectors of **E** and **H** are respectively obtained from the null spaces of



and



(Since usually **µ** = **I** the first equation is simpler.) The eigenvector components are given by the cofactors associated with any matrix row (no diagonalization is necessary!). Most generally, when one set of field eigenvectors is known, the other one can be obtained from

 or 

as follows from the general eigenequation for Tellegen CR. Note that solving the eigenproblem for the electric field and obtaining the magnetic field eigenvectors from the last relation amounts to generalizing the well-known Yeh’s 4×4 matrix method to optically active media.

The electric field eigenmatrix can likewise be cast into the form



where  is the *reduced gyration tensor* from the (complete) Born-Landau CR. (Conversely, .) The introduction of the gyration tensor is possible only if  that is, in the absence of (magnetic) dissipation. Since one further has



The above eigenmatrix provides the refractive indices and the electric field eigenvectors. As for the magnetic field eigenvectors,



These relations show that, if , then the Tellegen OA matrix **** is equivalent to Born-Landau reduced gyration tensor **g**’. Tellegen CR and complete Born-Landau CR are equivalent.

If the second-order effects in OA are neglected, the electric field eigenmatrix becomes



and can be put into correspondence with “classic” Born-Landau CR. Note that, to this degree of approximation, the equation of normals takes the particularly simple form



that is, the gyration term can be omitted, so that one essentially finds the refractive indices of the respective *optically inactive* medium. However, the electric field eigenvectors must be determined from the null space of the eigenmatrix with the gyration term included. (Eigenvectors are affected to the first order in the OA whereas eigenvalues are affected up to the second one.)

Similarly, if  that is, in the absence of absorption (non-dissipative medium), the magnetic field eigenmatrix can be parameterized as



where  is the *reduced magnetic gyration tensor* from the (complete) Bokut’- Serdyukov CR. (Conversely, .) Once the magnetic field eigenvectors obtained, the electric field ones can be determined from



Consequently, the Tellegen OA tensor **** can be replaced by the Bokut’ – Serdyukov reduced magnetic gyration tensor ; Tellegen CR and complete Bokut’ – Serdyukov CR are equivalent.

To the first in the OA, the magnetic field eigenmatrix takes the form



which corresponds to “classic” Bokut’ – Serdyukov CR. The equation of normals is

corresponding to the respective optically inactive medium. Of course, it has the same solutions as its electric field counterpart from the previous paragraph. The determination of the field eigenvectors involves however the gyration term. (Eigenvectors are affected to the first order in the OA whereas eigenvalues are affected up to the second one.)

1. *Transformed* Fedorov eigenmatrix (Gaussian upper-block elimination, applicable if ),



The eigenvalues and the eigenvectors for the electric field **E** *are invariant* by the transformation, so that they can be determined from the 3×3 eigenmatrix  as explained in the previous section. The equation of normals follows from the relation ,



provided that  and . The effective eigendimension reduces from 6 to 3.

Alternatively, by using the constitutive relation , one gets the following eigenmatrix for the electric induction,



Further, since  in virtue of Maxwell’s divergence equation (transversality of the electric induction) then  where  (decomposition into parallel and normal components of **D**) and the dimension for the equation of normals for the electric induction providing the refractive indices *n*1,2 can be further reduced to 2×2 *in the normal incidence case*,



provided that  and . The associated eigenvectors of **D** can be obtained from the null spaces of the eigenmatrices, as explained in the previous section. Knowing the eigenvectors of **D** one can then simply determine those of the electric field **E** from the constitutive relation

.

An analogous procedure can be applied to the magnetic induction **B**. If  one the has (Gaussian lower block elimination)



so that the magnetic fied eigenmatrix is



whereas the magnetic induction one is



in virtue of the constitutive relation . Since  (transversality of the magnetic induction) the equation of normals (from a 2×2 matrix *in the normal incidence case*)



provides the same refractive indices as the electric induction one (both sets of solutions coincide), whereas the eigenvectors of **B** are obtained from the null spaces of the eigenmatrices, as explained before. The eigenvectors of the magnetic field **H** are then simply obtained from the constitutive relation

.

Most generally, when one set of field eigenvectors is known, the other one can be obtained from



or

as follows from the general eigenequation for Fedorov CR. Note that solving the eigenproblem for the electric field and obtaining the magnetic field eigenvectors from the last relation amounts to generalizing the well-known Yeh’s 4×4 matrix method to optically active media.

Generally, the equations of normals contain inverses of matrices involving the refractive index *n*. This makes them of higher degree than those of Tellegen, Born – Landau and Bokut’s – Serdyukov CR.

The above methodology can likewise be applied to the rest of the block-diagonal CRs (complete Born – Landau and Bokut’ – Serdyukov).

1. *Transformed* complete Born – Landau eigenmatrix

Gaussian upper-block elimination effectively reducing the eigendimension from 6 to 3 (if |**µ**1| ≠ 0),



where  is the reduced gyration tensor (provided that ). The eigenmatrix (3×3) for the electric field **E** is thus,



or expressed in terms of the reduced gyration tensor **g**’ instead of ****1,



since . It can be easily shown that this eigenmatrix coincides with the electric field eigenmatrix from Tellegen CR.

Equation of normals (of third degree for *n*2 provided that and ),



Provides the refractive indices and the eigenvectors of **E** (coinciding with Fedorov ones; see next section). This equation of normals is more practical than Fedorov ones since it does feature inverses of matrices containing the refractive index.

If one considers OA *to the first order* only, one has for the electric field eigenmatrix



Note that, to this order of approximation, the permittivity and permeability tensors are those of Tellegen CR. Again to this degree of approximation, the equation of normals is simply



that is, the equation is solved *in the absence of OA*. These are the “classic” (approximate) Born – Landau CR.

The Born - Landau (both “classic” and complete) magnetic field eigenvectors are obtained from the relation



(follows from the general eigenequation). To transform to Tellegen - Fedorov magnetic-field eigenvectors use



(see next section). The relation turns out to be identical to that for Fedorov field eigenvectors, as expected. It can be likewise expressed in terms of the reduced gyration tensor only,



Consequently, one may effectively replace the Fedorov OA tensor ****1 by the Born-Landau reduced gyration tensor **g**’. Fedorov CR and complete Born – Landau CR are equivalent. “Classic” Born – Landau CR are a first-order approximation in OA.

1. *Transformed* complete Bokut’ – Serdyukov eigenmatrix

Gaussian lower block elimination effectively reducing the eigendimension from 6 to 3 (if |****1| ≠ 0),



where  is the reduced magnetic gyration tensor (provided that ).

The eigenmatrix (3×3) for the magnetic field **H** is thus,



or expressed in terms of the reduced magnetic gyration  tensor instead of ****1,



since . It can be easily shown that this eigenmatrix coincides with the magnetic field eigenmatrix from Tellegen CR.

Equation of normals (of third degree for *n*2 provided that  and ),



Provides the refractive indices and the eigenvectors of **H** (coinciding with Fedorov ones; see next section). This equation of normals is more practical than Fedorov ones since it does feature inverses of matrices containing the refractive index.

If one considers OA *to the first order* only, one has for the magnetic field eigenmatrix



Note that, to this order of approximation, the permittivity and permeability tensors are those of Tellegen CR. Again, to this degree of approximation, the equation of normals is simply



that is, the equation is solved *in the absence of OA*. These are the “classic” (approximate) Bokut’ - Serdyukov CR.

The Bokut’ - Serdyukov (both “classic” and complete) electric field eigenvectors are obtained from the relation



(follows from the general eigenequation). To transform to Tellegen-Fedorov electric-field eigenvectors use



(see next section). The relation turns out to be identical to that for Fedorov field eigenvectors, as expected. It can be likewise expressed in terms of the reduced magnetic gyration tensor only,



Consequently, one may effectively replace the Fedorov OA tensor ****1 by the Bokut’ - Serdyukov reduced magnetic gyration tensor **g**’. Fedorov CR and complete Bokut’ - Serdyukov CR are equivalent. “Classic” Bokut’ - Serdyukov CR are a first-order approximation in OA.

***Relations* *between eigenmatrices (relations between CRs)***

1. *Tellegen – Fedorov*: left multiplication by a suitable transformation matrix of the Tellegen eigenmatrix produces the Fedorov one (follows from Barkovsky – Borzdov paper),



This transformation eliminates the off-diagonal blocks from the constitutive tensor for Tellegen CR, but at the price of including curls into the permittivity and permeability tensors of Fedorov CR. The equivalence relations between the parameters of the two CRs follow immediately,



*The general equation of normals* (|…| = 0) *does not change* (since the determinant of the transition matrix never goes to zero.)

*The field eigenvectors do not change* (thanks to the left multiplication),



The inverse transition (Fedorov to Tellegen) is more complex,



wherefrom the inverse equivalence relations follow,



The inverse transformation eliminates the curls from the permittivity and the permeability tensors of Fedorov CR (originating from the Gibbs CR containing spatial derivatives), but creates off-diagonal blocks into the constitutive tensor for Tellegen CR.

1. *Fedorov – complete Born-Landau*, left- *and* right multiplications by two suitable transformation matrices of the Fedorov eigenmatrix produces the complete Born – Landau one (extension of Barkovsky – Borzdov paper),



by using the matrix identity  where  is the *reduced gyration vector* and  is the *gyration tensor* (**G** = **g u** is the *gyration vector*). Note that the introduction of the gyration tensor and vector is possible only if  that is, in the absence of absorption (non-dissipative medium).

Note that the two curl Maxwell equations are invariant with respect to the above matrix transformation (invariance of the two off-diagonal terms). This is a special case of a *Fedorov transformation*.

*The general equation of normals* (|…| = 0) *does not change* since



(the determinants in the fraction numerator and denominator never go to zero.)

*The electric field eigenvectors do not change however, the magnetic field eigenvectors**change* (because of the right multiplication) with respect to Fedorov – Tellegen ones,



The change of the magnetic field eigenvectors entails their special handling (see the previous paragraph). *The above relation between eigenvectors is valid for both complete and “classic” (approximate) Born – Landau CR.*

Neglecting the second-order OA terms in the complete Born – Landau CR results in the “classic” (approximate) ones,



1. Fedorov – complete Bokut’-Serdyukov. By using another *Fedorov transformation*, Bokut’-Serdyukov “magnetic” dual CR, including their “complete” version, can be derived from the Fedorov CR,



with the use of the matrix identity  where  is the *reduced magnetic gyration tensor* and is the *magnetic gyration tensor* (**G***m* = **g***m* **u** is the *magnetic gyration vector*). Note that the introduction of the gyration tensor and vector is possible only if  that is, in the absence of (magnetic) dissipation. These are the complete Bokut’s – Serdyukov CR.

*The general equation of normals* (|…| = 0) *does not change* since



(the determinants in the fraction numerator and denominator never go to zero.)

*The magnetic field eigenvectors do not change however, the electric field eigenvectors**change* (because of the right multiplication) with respect to Fedorov – Tellegen ones,



The change of the electric field eigenvectors entails their special handling (see the previous paragraph). *The above relation between eigenvectors is valid for both complete and “classic” (approximate) Bokut’s – Serdyukov CR.*

The complete – “classic” constitutive tensors coincide to the first order in the OA,



 THE END!

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